

ALGEBRAIC FUNCTIONS WITH FERMAT PROPERTY, EIGENVALUES OF TRANSFER OPERATOR AND RIEMANN ZEROS, AND OTHER OPEN PROBLEMS

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ABSTRACT. In this note we list a number of open problems in the fields of number theory, combinatorics, and representation theory: algebraic functions with Fermat property; power product expansion of the generating function for the partition function; relation between the non-trivial Riemann zeros and eigenvalues of the transfer operator; functional equation related to norm forms; two problems from geometric combinatorics; a problem on the moments of the Minkowski question mark function; a question in representation theory; a problem on interpolating the moments of the Stern's diatomic sequence; an arithmetic properties of the binary composition function.

My research in the next couple of years will be concentrated on projective superflows, transfer operator for the Gauss continued fraction map (also the Mayer-Ruelle operator), and the integration of the world of Minkowski question mark function with the world of modular and quasi-modular forms. Moreover, more than half of the time I have commitments in areas outside science, and this time will increase after these two years. So, here the list of a number of open problems of various levels of difficulty from various areas of mathematics is presented. Hopefully some of these problems will catch an attention of professionals, and I hope that colleagues can propose them to students in mathematics.

1. ALGEBRAIC FUNCTIONS WITH A FERMAT PROPERTY (number theory)

Consider the following rational function

$$P(x) = \frac{x^2}{(1-2x)(1-x)} = \sum_{n=2}^{\infty} s(n)x^n = \sum_{n=2}^{\infty} (2^{n-1} - 1)x^n.$$

Thus, we have $s(p) \equiv 0 \pmod{p}$ for $p > 2$ prime.

Next, let

$$J(x) = \frac{1}{\sqrt{1-4x}} - \frac{2}{1-x} = \sum_{n=0}^{\infty} s(n)x^n = \sum_{n=0}^{\infty} \left(\binom{2n}{n} - 2 \right) x^n.$$

This also gives $s(p) \equiv 0 \pmod{p}$ for $p \geq 2$ prime.

Finally, as is implied by ([5], Proposition 1), we have the same (for $p > 3$) conclusion for a degree 3 algebraic function $T(x) = 1 + x^2 + O(x^3)$, which satisfies

$$(6x^5 - 3x^4 + 2x^3 + 3x^2 - 1)T^3 + (x^5 - x^4 + x^3 + 2x^2 - 1)T^2 + (x^3 - x^2 + 1)T + 1 = 0.$$

Date: September 30, 2016.

The research of the author was supported by the Research Council of Lithuania grant No. MIP-072/2015.

Let $R(x) = \sum_{n=0}^{\infty} s(n)x^n \in \mathbb{Z}[[x]]$ be an algebraic function over $\mathbb{Q}(x)$, unramified at $x = 0$. Suppose, $s(p) \equiv 0 \pmod{p}$ for all sufficiently large prime numbers p . We call such a function $R(x)$ an *algebraic function with a Fermat property*. Thus, we formulate

Problem 1. *Characterize all algebraic functions with a Fermat property in general, and in any particular algebraic function field $\mathbb{Q}(x, U)$, where U is unramified at $x = 0$.*

We can multiply the function $R(x)$ by an integer to get the congruence valid for all primes. All algebraic functions with a Fermat property form an abelian group \mathcal{F} . Since the set of all algebraic functions over \mathbb{Q} is countable, \mathcal{F} is also countable. If $U(x) = \sum_{n=0}^{\infty} a(n)x^n \in \mathbb{Z}[[x]]$ is an algebraic function, then $U^\partial(x) := xU'(x) = \sum_{n=0}^{\infty} na(n)x^n \in \mathcal{F}$. Indeed, first it is obvious that $U^\partial \in \mathbb{Z}[[x]]$. And second, if $Q(Y, x) \in \mathbb{Z}[Y, x]$ and $Q(U, x) = 0$, then

$$U' = -\frac{Q_x(U, x)}{Q_Y(U, x)}$$

belongs to the same algebraic function field $\mathbb{Q}(U)$. Let D (from “Differential”) be the union of all such possible U^∂ . Then D is a subgroup of \mathcal{F} . So is $\mathbb{Z}[x]$. Finally, let $U(x) = \sum_{n=0}^{\infty} a(n)x^n \in \mathbb{Z}[[x]]$ is again an algebraic function, unramified and without a pole at $x = 0$, and let for an integer $M \geq 2$, $U^{(M)}(x) = U(x^M)$. Let P (from “Power”) be the group whose elements are

$$\sum_{j=1}^s U_j^{(M_j)}, \quad M_j \geq 2.$$

Then P is also a subgroup of \mathcal{F} . Of course, any two of the subgroups D , $\mathbb{Z}[x]$ and P have a non-trivial pairwise intersection. Let also for any U , algebraic over $\mathbb{Q}(x)$ and unramified at $x = 0$, $\mathcal{F}_U = \mathcal{F} \cap \mathbb{Q}(x, U)$, $D_U = D \cap \mathbb{Q}(x, U)$, $P_U = P \cap \mathbb{Q}(x, U)$ (We identify any algebraic function with its Laurent expansion at $x = 0$). We may refine Problem 1 as follows.

Problem 2. *Find the structure of abelian groups*

$$\mathcal{F}/(D + \mathbb{Z}[x] + P), \quad \mathfrak{A}_U = \mathcal{F}_U/(D_U + \mathbb{Z}[x]), \quad \mathfrak{P}_U = \mathcal{F}_U/(D_U + \mathbb{Z}[x] + P_U)$$

for any U algebraic over $\mathbb{Q}(x)$ and unramified at $x = 0$. In particular, for example, what is the group \mathfrak{P}_\emptyset (that is, we talk only about $\mathbb{Q}(x)$)?

In fact, for a function $J(x)$ we have an even stronger property $s(p) \equiv 0 \pmod{p^2}$. We may call it an *algebraic function with a strong Fermat (or Wieferich) property*, and ask similar questions.

2. POWER PRODUCT EXPANSION OF THE GENERATING FUNCTION FOR THE PARTITION FUNCTION (combinatorics, number theory)

This problem is extracted from [4]. Let $p(n)$ stand for the classical partition function. It is known that

$$P(x) = \sum_{n=0}^{\infty} p(n)x^n = \frac{1}{\prod_{k=1}^{\infty} (1 - x^k)}.$$

Consider now the power product expansion of $P(x)$. That is, define the unique sequence of integers n_k , $k \geq 1$, such that

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} (1 + n_k x^k).$$

This is the sequence A220420 in the Online Encyclopedia of Integer Sequences [12]. It is easily demonstrated [4] that $n_k = 1$ if k is odd, and for even indices, the sequence reads as

$$2, 4, 0, 14, -4, -8, -16, 196, -54, -92, -184, 144, -628, -1040, -2160, 41102, \dots$$

Looking at these terms, we may ask the following.

Problem 3. *Explore the arithmetic properties of the sequence n_k . In particular, is it true that*

$$n(8k+2), n(8k+4), n(8k+6) \text{ are negative, } n(8k) \text{ are positive for } k \geq 1.$$

This is probably easy (if we use the recurrence given in [4]), but deeper properties of the sequence $\{n_k : n \in \mathbb{N}\}$ wait to be discovered.

3. MAP FROM SET OF NON-TRIVIAL ZEROS OF THE RIEMANN ZETA FUNCTION TO THE SET OF EIGENVALUES OF THE TRANSFER OPERATOR FOR THE GAUSS CONTINUED FRACTION MAP (analytic number theory)

Let \mathbb{D} be the disc $\{z \in \mathbb{C} : |z - 1| < \frac{3}{2}\}$. Let \mathbf{V} be the Banach space of functions which are analytic in \mathbb{D} and are continuous in its closure, with the supremum norm. For a complex number s , $\Re(s) > \frac{1}{2}$, one defines the Mayer-Ruelle transfer operator by [10]

$$L_s[f(t)](z) = \sum_{m=1}^{\infty} \frac{1}{(z+m)^{2s}} f\left(\frac{1}{z+m}\right).$$

The operator L_s is extended to all complex numbers s by an analytic continuation. As the fundamental contribution, it was proved by D. Mayer that [10]

$$\det(1 - L_s^2) = \prod_{n=1}^{\infty} (1 - \lambda_n^2(s)) = Z(s),$$

where on the left we have the Fredholm determinant of the operator defined as the product involving eigenvalues $\lambda_n(s)$ of the operator L_s (as given in the middle), and on the right - the Selberg zeta function for the full modular group.

It is known that one subset of non-trivial zeros of $Z(s)$ are given by $\frac{\rho}{2}$, where ρ are non-trivial zeros of the Riemann zeta function [9]. It was proved in [3] (the project is being planned to be finished by March 2017), that the labelling of eigenvalues λ_n with an integer n is canonical, corresponding to polynomials of degree $(n+1)$: this ordering corresponds exactly to ordering λ_n according to their magnitude. In particular, for each nontrivial zero ρ of the Riemann zeta function there exist an integer $M = t(\rho)$ such that

$$\lambda_M\left(\frac{\rho}{2}\right) = 1.$$

See [9] to see that this number is 1, which corresponds to Riemann zeros and even Maass wave forms, not -1 , which corresponds to odd Maass wave forms. For example, if we order nontrivial zeros according to the magnitude of imaginary part, calculations show that

$$t(\rho_1) = 1, \quad t(\rho_2) = 2, \quad t(\rho_3) = 1, \quad t(\rho_4) = 3, \quad t(\rho_5) = 1, \quad t(\rho_6) = 3.$$

In particular, we pose

Problem 4. *Given an integer $M \in \mathbb{N}$. What one can say about the set $t^{-1}(M)$? Is it infinite? How the conjectural distribution of Riemann zeros change if we limit ourselves to the set $t^{-1}(M)$? At least, how one can compute, for example, the set $t^{-1}(1)$ effectively?*

The question about trivial zeros of $Z(s)$ seems also interesting. Let $k \in \mathbb{N}$. It is known that order of vanishing at $s = 1 - k$ of $Z(s)$ equals the dimension of the corresponding space of cusp forms M_{2k} for $\mathrm{PSL}_2(\mathbb{Z})$. In particular,

Problem 5. *How the number $\dim_{\mathbb{C}}(M_{2k})$ distributes among different factors of*

$$Z(1 - k) = \prod_{n=1}^{\infty} \left(1 - \lambda_n^2(1 - k)\right).$$

We re-iterate that the crucial thing here that the index n is attributed canonically.

4. PSEUDO-AUTOMORPHISMS OF NORM FORMS IN NUMBER FIELDS (**algebraic number theory, functional equations**)

This problem is taken from [2].

Problem 6. *Let $T(a_1, a_2, \dots, a_n)$ be a norm form in some integral basis of some proper field extension of \mathbb{Q} of degree n . Find all functions $f : \mathbb{Z} \rightarrow \mathbb{C}$, such that*

$$T(f(a_1), f(a_2), \dots, f(a_n)) \text{ depends only on the value of } T(a_1, a_2, \dots, a_n).$$

This problem was solved by U. Zannier in case we have a norm form $a^2 + b^2$. In essence, there are 6 independent solutions, given by a constant, a square function, and characters of orders 2, 3, 4, 5 (see [2]).

In particular, consider the polynomial $h(X) = X^3 - 3X + 1$. Let α be any root of it. Then the splitting field of this polynomial is cubic, given by $\mathbf{K} = \mathbb{Q}(\alpha)$. Let $\alpha' = \alpha^2 - 2$. The set $\{1, \alpha, \alpha'\}$ is an integral basis of the ring of integers, and the norm form is

$$T(a, b, c) = \mathcal{N}(a + b\alpha + c\alpha') = a^3 - b^3 - c^3 - 3ab^2 - 3ac^2 + 3abc + 6b^2c - 3bc^2.$$

Problem 7. *Find all functions $f, h : \mathbb{Z} \rightarrow \mathbb{C}$, such that*

$$T(f(a), f(b), f(c)) = h(T(a, b, c)).$$

In [2] it is shown that if $f \equiv h$, then $f(n) = n$, $f(n) = -n$, $f(n) \equiv 0$, $f(n) \equiv i$, or $f(n) \equiv -i$. But it is much more natural to pose a problem with $f \neq h$, since this definitely contains much more information on the arithmetic of the field.

5. FRIENDLY PATHS (geometric combinatorics)

Here I reproduce my problem from American Mathematical Monthly [6], Problem 11484, which still has no solution.

An *uphill* lattice path is the union of a (doubly infinite) sequence of directed line segments in \mathbb{R}^2 , each connecting an integer pair (a, b) to an adjacent pair either $(a, b + 1)$, $(a + 1, b)$. A *downhill* lattice path is defined similarly, but with $b - 1$ in place of $b + 1$, and a *monotone* lattice path is an uphill or downhill lattice path.

Given a finite set P of points in \mathbb{Z} , a *friendly* path is a monotone lattice path for which there are as many points in P on one side of the path as on the other. (Points that lie on the path do not count.)

Problem 8. *Is it true that for every odd-sized set of points there is a friendly path?*

6. LINES CROSSING ALL SQUARES OF THE BOARD (geometric combinatorics)

This is my problem from c. 1998. It is so natural, that possibly it might had occurred to other people as well.

Let $n \in \mathbb{Z}$. Given a square $n \times n$, divided into n^2 unit squares in the usual way. A line is said to *intersect* a unit square, if they have a common point, which is an inner point of the unit square.

Problem 9. *Find the minimal number of lines, such that each of n^2 unit squares is intersected. In particular, is it true that for $n \geq 3$ this number is equal to $n - 1$?*

7. FORMULA FOR THE MOMENTS OF THE MINKOWSKI QUESTION MARK FUNCTION (complex dynamics, number theory)

This problem is taken from [1], where it is posed as a Conjecture with a positive answer.

The *Minkowski question mark function* $?(x)$ is defined by

$$?([0, a_1, a_2, a_3, \dots]) = 2^{1-a_1} - 2^{1-(a_1+a_2)} + 2^{1-(a_1+a_2+a_3)} - \dots, \quad x \in [0, 1],$$

where $x = [0, a_1, a_2, \dots]$ stands for the representation of x by a regular continued fraction.

Now, define the rational functions $\mathbf{Q}_n(z) = \mathcal{B}_n(z)z^{-n-1}$, $n \geq 0$, (where $\mathcal{B}_n(z)$ are polynomials with rational coefficients, which are p -adic integers for $p \neq 2$) by

$$\mathbf{Q}_0(z) = -\frac{1}{2z}, \text{ and recurrently by } \mathbf{Q}_n(z) = \frac{1}{2} \sum_{j=0}^{n-1} \frac{1}{j!} \cdot \frac{d^j}{dz^j} \mathbf{Q}_{n-j-1}(-1) \cdot \left(z^j - \frac{1}{z^{j+2}} \right).$$

The sequence $\mathbf{Q}'_n(-1) = \frac{d}{dz} \mathbf{Q}_n(z)|_{z=-1}$, $n \in \mathbb{N}_0$, begins with $\frac{1}{2}, -\frac{1}{2}, 1, -\frac{5}{2}, \frac{25}{4}, -16, 43, -\frac{971}{8}, \frac{1417}{4}, \dots$

Problem 10. *Is the following true? The function $\Lambda(t) = \sum_{n=0}^{\infty} \frac{\mathbf{Q}'_n(-1)}{n!} t^n$ is an entire function, and*

$$\int_0^{\infty} \Lambda(t) e^{-t} dt = \int_0^1 x^2 d?(x).$$

The same can be asked about higher moments, which are encoded by the same rational functions \mathbf{Q}_n ; see [1].

8. SUPERFLOWS WITH THE SMALLEST POSSIBLE SYMMETRY GROUP (group representations, algebra)

We give a definition of the superflow, but in fact the question itself is purely algebraic from the theory of representation of finite groups over \mathbb{C} .

Definition 1. *Let $n \in \mathbb{N}$, $n \geq 2$, and $\Gamma \hookrightarrow \mathrm{GL}(n, \mathbb{R})$ be an exact representation of a finite group, and we identify Γ with the image. We call the flow $\phi(\mathbf{x})$ the Γ -superflow, if*

- i) *there exists a vector field $\mathbf{Q}(\mathbf{x}) = (Q_1, \dots, Q_n) \neq (0, \dots, 0)$ whose components are 2-homogenic rational functions and which is exactly the vector field of the flow $\phi(\mathbf{x})$, such that*

$$\gamma^{-1} \circ \mathbf{Q} \circ \gamma(\mathbf{x}) = \mathbf{Q}(\mathbf{x}) \tag{1}$$

is satisfied for all $\gamma \in \Gamma$, and

- ii) *every other vector field \mathbf{Q}' which satisfies (1) for all $\gamma \in \Gamma$ is either a scalar multiple of \mathbf{Q} , or its degree of a common denominator is higher than that of \mathbf{Q} .*

The superflow is said to be reducible or irreducible, if the representation $\Gamma \hookrightarrow \mathrm{GL}(n, \mathbb{R})$ (considered as a complex representation) is reducible or, respectively, irreducible.

Let $n \geq 2$ be an integer, and let

$$\psi_n = \inf_{\Gamma} \frac{|\Gamma|}{n!};$$

the infimum is taken over all finite subgroups of $\mathrm{GL}(n, \mathbb{R})$ for which there exist a superflow. It was proved in [7] that for $n \geq 3$, one has $\psi_n \leq 2$.

Problem 11. *Is it true that for $n \geq 3$, one has $\psi_n = 2$?*

9. FUNCTION WHOSE MOMENTS ARE MOMENTS OF THE STERN'S DIATOMIC SEQUENCE (number theory, calculus)

Let $a(n)$ stand for the Stern's diatomic sequence: $a(0) = 0$, $a(1) = 1$, and for $n \geq 1$, one has $a(2n) = a(n)$, $a(2n+1) = a(n) + a(n+1)$.

Let us define the “moments”

$$Q^{(L)}(N) = \sum_{n=2^{N+1}}^{2^{N+1}+1} a^L(n), \quad L \in \mathbb{N}_0.$$

1. Sequence μ_L			
L	μ_L	Minimal polynomial of $2\mu_L$	G
0	1	$\lambda - 2$	Z_1
1	$3/2$	$\lambda - 3$	Z_1
2	$\frac{1}{4}(5 + \sqrt{17})$	$\lambda^2 - 5\lambda + 2$	Z_2
3	$7/2$	$\lambda - 7$	Z_1
4	$\frac{1}{4}(11 + \sqrt{113})$	$\lambda^2 - 11\lambda + 2$	Z_2
5	$\frac{1}{2}(7 + 4\sqrt{6})$	$\lambda^2 - 14\lambda - 47$	Z_2
6	$\frac{1}{4}(10 + \sqrt{265} + \sqrt{357 + 20\sqrt{265}})$	$\lambda^4 - 20\lambda^3 - 161\lambda^2 - 40\lambda + 4$	D_4
7	20.50916706...	$\lambda^3 - 29\lambda^2 - 485\lambda - 327$	S_3
8	$\frac{1}{4}(22 + \sqrt{1801} + \sqrt{2277 + 44\sqrt{1801}})$	$\lambda^4 - 44\lambda^3 - 1313\lambda^2 - 88\lambda + 4$	D_4
9	50.69978074...	$\lambda^3 - 65\lambda^2 - 3653\lambda - 3843$	S_3
10	$\frac{1}{4}(50 + 3\sqrt{1345} + \sqrt{14597 + 300\sqrt{1345}})$	$\lambda^4 - 100\lambda^3 - 9601\lambda^2 - 200\lambda + 4$	D_4
11	126.5114484...	$\lambda^4 - 156\lambda^3 - 24882\lambda^2 + 83828\lambda + 107529$	S_4
12	200.4256707...	$\lambda^6 - 247\lambda^5 - 63659\lambda^4 + 797003\lambda^3...$	$Z_2 \wr S_3$

TABLE 1. The moments of the Stern's diatomic sequence, minimal polynomials of doubled moments, and Galois groups. Z - cyclic, D - dihedral, S - symmetric.

It is known (and easily provable) that $Q^{(1)}(N) = \sum_{n=2^{N+1}}^{2^{N+1}} a(n) = 3^N$. Also [11],

$$Q^{(2)}(N) = \frac{1}{\sqrt{17}} \left[\left(\frac{5 + \sqrt{17}}{2} \right)^{N+1} - \left(\frac{5 - \sqrt{17}}{2} \right)^{N+1} \right].$$

Proposition 1. *Sequence $Q^{(L)}(N)$, $N \in \mathbb{N}$, satisfies the linear recurrence of degree L , and there exists a unique algebraic number μ_L (half of an algebraic integer) and a constant c_L , such that*

$$\frac{Q^{(L)}(N)}{2^N} \sim c_L \mu_L^N.$$

So, from above, $\mu_0 = 1$, $\mu_1 = \frac{3}{2}$, $\mu_2 = \frac{5+\sqrt{17}}{4}$. The Table 1 gives information on the initial values of μ_L . It is also known that the sequence $\mu_L^{1/L}$ is increasing, bounded, and

$$\lim_{L \rightarrow \infty} \mu_L^{1/L} = \frac{1 + \sqrt{5}}{2}.$$

Problem 12. *Does there exist a function $\mathbf{p}(x)$ (analytic? continuous of bounded variation? generalized - that is, a distribution?) such that*

$$\int_0^{\frac{1+\sqrt{5}}{2}} x^L d\mathbf{p}(x) = \mu_L, \quad L \geq 0.$$

If it does, this function has algebraic moments, and encodes a lot of structural information about continued fractions.

10. BINARY COMPOSITION FUNCTION

Let $\vartheta(n)$ denote the number of compositions (ordered partitions) of a positive integer into powers of 2. This is the sequence A023359 in The Online Encyclopedia of Integer Sequences [12]. The generating function satisfies

$$\sum_{n=0}^{\infty} \vartheta(n)x^n = \frac{1}{1 - \sum_{k=0}^{\infty} x^{2^k}}.$$

It was proved in [8] that for any integer a (positive, zero or negative) there exist a limit in a 2-adic topology

$$\lim_{k \rightarrow \infty} \vartheta(2^k + a) = \Theta(a) \in \mathbb{Z}_2,$$

where the latter stands for the ring of integers of the 2-adic number field \mathbb{Q}_2 . For example,

$$\Theta(0) = 2^3 + 2^8 + 2^9 + \dots.$$

Problem 13. Describe arithmetic properties of numbers $\Theta(a)$. Is it rational or algebraic over \mathbb{Q} ?

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